4-1. Equation and polynomials of Laguerre
We define Laguerre differential equation in the following form
\begin{equation}
\frac{d^2 y}{dx^2} + (1 - x) \frac{dy}{dx} + ny = 0
\end{equation}
And its solution is
\begin{equation}
L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r}{(n-r)!(r!)^2} x^r
\end{equation}

- Generating function
Theorem 1.
\begin{equation}
\frac{\exp \{-xt/(1-t)\}}{1-t} = \sum_{n=0}^\infty L_n(x)t^n
\end{equation}
Proof. Since
\begin{align*}
\frac{1}{1-t} \exp \{-xt/(1-t)\} &= \frac{1}{1-t} \sum_{r=0}^\infty \frac{1}{r!} (-\frac{xt}{1-t})^r \\
&= \sum_{r=0}^\infty \frac{(-1)^r}{r!(1-t)^{r+1}} x^r t^r \\
\end{align*}
From Polynomials we have
\begin{align*}
\frac{1}{(1-t)^{r+1}} &= 1 + (r+1)t + \frac{(r+1)(r+2)}{2!} t^2 + \frac{(r+1)(r+2)(r+3)}{3!} t^3 + \ldots \\
&= \sum_{k=0}^\infty \frac{(r+k)!}{k!r!} t^k
\end{align*}
Then we have
\begin{equation}
\frac{1}{1-t} \exp \{-xt/(1-t)\} = \sum_{r,k=0}^\infty (-1)^r \frac{(r+k)!}{(r!)^2k!} x^r t^{r+k}
\end{equation}
For fixed \( r \) we can get the coefficient of \( t^n \) by putting \( r+k = n \), then coef. of \( t^n \) in equation (4.8) as following
\begin{align*}
(-1)^r \frac{n!}{(r!)^2(n-r)!} x^r
\end{align*}
Then we have the following relation:
\begin{equation}
L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(r!)^2(n-r)!} x^r.
\end{equation}
• **Important relation for Laguerre polynomials**

**Theorem 2.**

\[(4.10) \quad L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x})\]

Use Leibnitz’s theorem for differentiation

\[
\frac{d^n}{dx^n} (uv) = \sum_{r=0}^{n} \frac{n!}{(n-r)! r!} \frac{d^{n-r}}{dx^{n-r}} u \left( \frac{d^r}{dx^r} v \right).
\]

We have the following

\[
\frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) = \frac{e^x}{n!} \sum_{r=0}^{n} \frac{n!}{(n-r)! r!} \frac{d^{n-r}}{dx^{n-r}} x^n \left( \frac{d^r}{dx^r} e^{-x} \right)
\]

But

\[
\frac{d^k}{dx^k} x^m = m(m-1) \cdots (m-k+1)x^{m-k}
\]

Then we have the following:

\[
\frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) = \frac{e^x}{n!} \sum_{r=0}^{n} \frac{n!}{(n-r)! r!} x^r (-1)^r e^{-x}
\]

\[
= \sum_{r=0}^{n} (-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r = L_n(x)
\]

Then form Laguerre function we can deduce that:

\[
L_0(x) = 1 \quad , \quad L_1(x) = 1 - x
\]

\[
L_2(x) = \frac{1}{2!} (x^2 - 4x + 2)
\]

\[
L_3(x) = \frac{1}{3!} (-x^3 + 9x^2 - 18x + 6)
\]

\[
L_4(x) = \frac{1}{4!} (x^4 - 16x^3 + 72x^2 - 96x + 24)
\]

**Theorem .r**

(i) \( L_n(0) = 1 \) \quad (ii) \( L'_n(0) = -n \)

**Proof.** Since

\[
\frac{1}{1-t} \exp \left\{ -xt/(1-t) \right\} = \sum_{n=0}^{\infty} L_n(x)t^n
\]

Put \( x = 0 \), we have

\[
\frac{1}{1-t} = \sum_{n=0}^{\infty} L_n(0)t^n
\]
Using the binomial theorem in the L.H.S. we have
\[
\frac{1}{1-t} = \sum_{m=0}^{\infty} t^m
\]
Then we have
\[
\sum_{m=0}^{\infty} t^m = \sum_{n=0}^{\infty} L_n(t)
\]
Equating coef. of \( t^n \) we get relation (i).
To have relation (ii) we use General Laguerre equation
\[
x \frac{d^2}{dx^2} L_n(x) + (1-x) \frac{d}{dx} L_n(x) + nL_n(x) = 0
\]
Put \( x = 0 \) we have
\[
L_n'(0) + nL_n(0) = 0
\]
Use relation (i) we have
\[
L_n'(0) = -n
\]
which complete the proof.

Similarly as in the above theorem, we can prove the following relations
\[
L_n''(0) = \frac{1}{2} n(n-1)
\]
Differentiating the generating function twice we have the following relation
\[
\frac{1}{1-t} e^{\frac{-t}{1-t}} \left( \frac{-t}{1-t} \right)^2 = \sum_{n=0}^{\infty} L_n''(x) t^n
\]
Put \( x = 0 \) we have
\[
\frac{t^2}{(1-t)^3} = \sum_{n=0}^{\infty} L_n''(0) t^n
\]
Use the binomial theorem in the L.H. S. we have
\[
t^2 \left\{ 1 + 3t + \frac{3 \cdot 4}{2!} t^2 + \frac{3 \cdot 4 \cdot 5}{3!} t^3 + \cdots + \frac{3 \cdot 4 \cdot 5 \cdots n}{(n-2)!} t^{n-2} + \cdots \right\} = \sum_{n=0}^{\infty} L_n''(0) t^n
\]
Equating coef. of \( t^n \), we have
\[
L_n''(0) = \frac{3 \cdot 4 \cdot 5 \cdots n}{(n-2)!} = \frac{n!}{2(n-2)!} = \frac{n(n-1)}{2}
\]

Orthogonal relation for Laguerre Polynomials.

Theorem 4.
\[
(4.12) \int_{0}^{\infty} e^{-x} L_n(x) L_m(x) dx = \delta_{nm}
\]
Proof. From the generating function we have
\[
\frac{\exp \{ -xt / (1-t) \}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n \\
\frac{\exp \{ -xs / (1-s) \}}{1-s} = \sum_{m=0}^{\infty} L_m(x) s^m
\]

Then we have
\[
\frac{1}{(1-t)(1-s)} \exp \left\{ -\frac{xt}{1-t} \right\} \exp \left\{ -\frac{xs}{1-s} \right\} = \sum_{n,m} L_n(x) L_m(x) t^n s^m
\]

Multiply both sides by \( e^{-x} \) and integrate we have
\[
(4.13) \int_{0}^{\infty} e^{-x} L_n(x) L_m(x) dx = I
\]

Then we claim
\[
\int_{0}^{\infty} e^{-x} L_n(x) L_m(x) dx
\]

Represent the coef. of \( t^n s^m \) in the integral \( I \), where
\[
I = \int_{0}^{\infty} e^{-x} \exp \left\{ -\frac{xt}{1-t} \right\} \exp \left\{ -\frac{xs}{1-s} \right\} dx
\]
\[
= \frac{1}{(1-t)(1-s)} \int_{0}^{\infty} \exp[-x] \cdot \exp\left[-\frac{xt}{1-t}\right] \cdot \exp\left[-\frac{xs}{1-s}\right] dx
\]
\[
= \frac{1}{(1-t)(1-s)} \int_{0}^{\infty} \exp \left\{ -x \left( 1 + \frac{t}{1-t} + \frac{s}{1-s} \right) \right\} dx
\]
\[
= \frac{1}{(1-t)(1-s)} \left[ \frac{-1}{1 + \{ t/(1-t) \} + \{ s/(1-s) \} } \exp \left\{ -x \left( 1 + \frac{t}{1-t} + \frac{s}{1-s} \right) \right\} \right]_{0}^{\infty}
\]
\[
= \frac{1}{(1-t)(1-s)} \cdot \frac{1}{1 + \frac{t}{1-t} + \frac{s}{1-s}}
\]
\[
= \frac{1}{(1-t)(1-s) + t(1-s) + s(1-t)}
\]
\[
= \frac{1}{1-st}
\]

Then we have the value of the integral
\[
I = \sum_{n=0}^{\infty} s^n t^n
\]

Then relation (4.13) becomes
\[ t^n s^m \int_0^\infty \! e^{-x} L_n(x) L_m(x) \, dx = t^n s^n \]

Then if \( n \neq m \) we have

\[ (4.14) \int_0^\infty \! e^{-x} L_n(x) L_m(x) \, dx = 0 \]

If \( n = m \), then

\[ (4.15) \int_0^\infty \! e^{-x} \left( L_n(x) \right)^2 \, dx = 1 \]

We can represent any relation in Laguerre polynomial by using this relation as following. Let \( f(x) \) is defined for all values of \( x \) then we can write it in the form

\[ f(x) = \sum_{n=0}^\infty A_n L_n(x) \]

Apply orthogonal relation, we have

\[ (4.16) A_n = \int_0^\infty \! e^{-x} f(x) L_n(x) \, dx . \]

### 4.7. Recurrence relations.

We will prove some important recurrence relations as following

\[ (4.17) \]

(i) \( (n + 1) L_{n+1}(x) = (2n + 1 - x) L_n(x) - nL_{n-1}(x) \)

**Proof.** Since

\[ \frac{1}{1-t} \exp \left\{ -xt/(1-t) \right\} = \sum_{n=0}^\infty L_n(x) t^n \]

Differentiate w.r.to \( t \), we have

\[ \frac{1}{(1-t)^2} \left\{ -x(1-t) \frac{(1-t)+t}{(1-t)^2} \exp \left\{ \frac{-xt}{1-t} \right\} + \exp \left\{ \frac{-xt}{1-t} \right\} \right\} \]

\[ = \sum_{n=1}^\infty \frac{L_n(x)}{(n-1)!} t^{n-1} \]

Then we have

\[ \frac{1}{(1-t)^3} \left\{ -x \exp \left\{ \frac{-xt}{1-t} \right\} + (1-t) \exp \left\{ \frac{-xt}{1-t} \right\} \right\} = \sum_{n=1}^\infty nL_n(x) t^{n-1} \]

\[ \frac{1-x-t}{(1-t)^3} \exp \left\{ \frac{-xt}{1-t} \right\} = \sum_{n=1}^\infty nL_n(x) t^{n-1} \]

The L.H.S. in the last equation can be written in the form

\[ \frac{1-x-t}{(1-t)^2} \sum_{n=0}^\infty L_n(x) t^n = \sum_{n=1}^\infty nL_n(x) t^{n-1} \]
Multiply both sides by \((1-t)^2\), we have
\[
(1-x)\sum_{n=0}^{\infty} L_n(x)t^n - \sum_{n=0}^{\infty} L_n(x)t^{n+1} = (1-2t+t^2)\sum_{n=1}^{\infty} nL_n(x)t^{n-1}
\]

Equating coeff. of \(t^n\), we have
\[
(1-x)L_n(x) - nL_{n-1}(x) = (n+1)L_{n+1}(x) - 2nL_n(x) + (n-1)L_{n-1}(x)
\]
Then
\[
(1-x+2n)L_n(x) - nL_{n-1}(x) = (n+1)L_{n+1}(x)
\]
Which complete the proof.

To prove the relation
\[(4.18)\] (ii) \(L'_n(x) = [L'_{n-1}(x) - L_{n-1}(x)]\)

Differentiate the generating function (*) w.r.to \(x\) we have
\[
\frac{-t}{(1-t)^2} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L'_n(x)t^n
\]
OR
\[
(\star\star) -\frac{t}{1-t} \sum_{n=0}^{\infty} L_n(x)t^n = \sum_{n=0}^{\infty} L'_n(x)t^n
\]

Multiply both sides by \((1-t)\), we have
\[
-\sum_{n=0}^{\infty} L_n(x)t^{n+1} = \sum_{n=0}^{\infty} L'_n(x)t^n - \sum_{n=0}^{\infty} L'_n(x)t^{n+1}
\]
Equating coeff. of \(t^n\) in both sides
\[
-L_{n-1}(x) = L'_n(x) - L'_{n-1}(x)
\]
Then we have
\[
[L'_{n-1}(x) - L_{n-1}(x)] = L'_n(x)
\]
We can obtain many recurrence relations from equation (4.18) by shifting \(n\) by \(n-1\).

**We can prove the following relation:**

\[(4.19)\] (iii) \(xL'_n(x) = nL_n(x) - nL_{n-1}(x)\)

To prove this relation use relation (4.17)
\[
(1+n)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)
\]
Differentiate w.r.to \(x\), we have
\[(4.20)\] \((1+n)L'_{n+1}(x) = (2n+1-x)L'_n(x) - L_n(x) - nL'_{n-1}(x)\]
And use the relation
\[
L'_{n+1}(x) = L'_n(x) - L_n(x)
\]
And the following form
\[ L'_{n-1}(x) = L'_n(x) + L_{n-1}(x) \]

We get the following:
\[
(n + 1)\left\{ L'_n(x) - L_n(x) \right\} = (2n + 1 - x) L'_n(x) - L_n(x) - n\left\{ L'_n(x) - L_{n-1}(x) \right\}
\]

Then we have
\[-nL_n(x) = -xL'_n(x) - nL_{n-1}(x)\]

Then we have the required.

**To prove the following relation**

\[
(4.21) \text{ (iv) } L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)
\]

Where
\[
(i) \frac{1}{1-t} = \sum_{r=0}^{\infty} t^r
\]

From the relation (***)
\[
\sum_{n=0}^{\infty} L'_n(x)t^n = \frac{-t}{1-t} \sum_{n=0}^{\infty} L_n(x)t^n
\]

Use relation (i) in the relation (***) , we have
\[
\sum_{n=0}^{\infty} L'_n(x)t^n = -t \sum_{s=0}^{\infty} t^s \sum_{s=0}^{\infty} L_s(x) t^s
\]

Which can be written in the form
\[
\sum_{n=0}^{\infty} L'_n(x)t^n = -\sum_{r,s=0}^{\infty} L_s(x) t^{r+s+1}
\]

To have coeff. of \( t^n \), we should put in the R.H.S. \( n = r + s + 1 \) or \( r = n - s - 1 \), then
\[ n - s - 1 \geq 0 \text{ or } s \leq n - 1, \text{ we have} \]
\[
\text{coeff. } t^n = \sum_{s=0}^{n-1} -L_s(x)
\]

Then we have
\[
L'_n(x) = -\sum_{s=0}^{n-1} L_s(x).
\]

**f-V. Associated Laguerre Polynomial**

The differential equation
\[(4.22) x \frac{d^2 y}{dx^2} + (k + 1 - x) \frac{dy}{dx} + ny = 0\]

Is called associated laguerre equation. If \( k = 0 \) is called Laguerre equation and its solution is \( y = L_n(x) \), while the solution of equation (4.22) is from the following theorem.
**Theorem 5.** If \( z(x,s) \) is solution of Laguerre equation of order \((n+k)\), then 
\[
d^k z \frac{d^k}{dx^k} \]
 satisfies associated Laguerre equation.

**Proof.** Since \( z(x,s) \) is a solution of Laguerre equation of order \((n+k)\), then it satisfy the equation
\[
(4.23) x \frac{d^2 z}{dx^2} + (1-x) \frac{dz}{dx} + (n+k)z = 0
\]
Differentiate \(k\) times and use Leibniz theorem as following
\[
\left[ x \frac{d^2 z}{dx^2} \right]^{(k)} + \left[ (1-x) \frac{dz}{dx} \right]^{(k)} + (n+k)z^{(k)} = 0
\]
Where \((k)\) represent the derivative of order \(k\).
Then we have
\[
x z^{(k+2)} + k z^{(k+1)} + (1-x) z^{(k+1)} - k z^{(k)} + (n+k) z^{(k)} = 0
\]
This can be written in the following form:
\[
x z^{(k+2)} + (k+1-x) z^{(k+1)} + nz^{(k)} = 0
\]
The last equation can be written in the following form:
\[
x \frac{d^2 z}{dx^2} z^{(k)} + (k+1-x) \frac{dz}{dx} z^{(k)} + n z^{(k)} = 0
\]
Then \(z^{(k)}\) satisfy the required equation.

From the above it is clear that \( L_n(x) \) satisfy Laguerre equation. Then
\[
\frac{d^k}{dx^k} L_{n+k}(x) \text{ satisfy associated Laguerre equation, then from the definition}
\]
\[
(4.24) L_n^k(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x) \quad (k < n)
\]
This is called associated Laguerre function.

**Theorem 6.**
\[
(4.25) L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{(n+k)! x^r}{(n-r)!(k+r)! r!}
\]

**Proof.** Since we proved that
\[
L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n! x^r}{(n-r)!(r!)^2} \quad \text{By shifting every } n \text{ by } n+k, \text{ we have}
\]
\[
(4.26) L_{n+k}(x) = \sum_{r=0}^{n+k} (-1)^r \frac{(n+k)! x^r}{(n+k-r)!(r!)^2}
\]
Use (4.24), we have
\[ L^k_n(x) = (-1)^k \frac{d^k}{dx^k} \sum_{r=0}^{n+k} (-1)^r \frac{(n+k)!x^r}{(n+k-r)!(r!)^2} \]

Note that the derivatives equal zero for \( r < k \), then the last equation can be written in the form:

\[ L^k_n(x) = (-1)^k \frac{d^k}{dx^k} \sum_{r=k}^{n+k} (-1)^r \frac{(n+k)!x^r}{(n+k-r)!(r!)^2} \]

Note that:

\[
\frac{d^k}{dx^k} x^r = r(r-1)\cdots(r-k+1)x^{r-k}
\]

\[ = x^{r-k} \left[ \frac{r(r-1)\cdots(r-k+1)(r-k)(r-k-1)\cdots3\cdot2\cdot1}{(r-k)(r-k-1)\cdots3\cdot2\cdot1} \right]
\]

\[ = \frac{r!}{(r-k)!} x^{r-k} \]

Then we have

\[ L^k_n(x) = (-1)^k \sum_{r=k}^{n+k} (-1)^r \frac{(n+k)!x^{r-k}}{(n+k-r)!(r!)^2} \]

Put \( r-k = s \), we have

\[ L^k_n(x) = (-1)^k \sum_{s=0}^{n} (-1)^{k+s} \frac{(n+k)!x^s}{(n-s)!s!(s+k)!} \]

And we have

\[ L^k_n(x) = \sum_{s=0}^{n} (-1)^s \frac{(n+k)!x^s}{(n-s)!(s+k)!s!} \]

Then we can prove the following simple relations:

\[ L'_1(x) = -1 \quad , \quad L'_2(x) = -4 + 2x \quad , \quad L'_3(x) = 2 \cdot \]

4-7 Properties of associated Laguerre function:

**Theorem 4.** The generating function for associated Laguerre function takes the form:

\[ (4.27) \quad \frac{\exp \{-xt/(1-t)\}}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} L^k_n(x)t^n. \]

**Proof.** Since

\[ \frac{\exp \{-xt/(1-t)\}}{(1-t)} = \sum_{n=0}^{\infty} L_n(x)t^n \]

Differentiate \( k \) times, we have
\[ \frac{d^k}{dx^k} \left[ \exp \left\{ -xt/(1-t) \right\} \right] = \frac{d^k}{dx^k} \sum_{n=0}^{\infty} L_n(x)t^n \quad (k < n) \]

Since all the derivative of order less than \( k \) equal to zero, then the last equation can be written as

\[ (\ast) \frac{d^k}{dx^k} \left[ \exp \left\{ -xt/(1-t) \right\} \right] = \frac{d^k}{dx^k} \sum_{n=k}^{\infty} L_n(x)t^n \]

Put \( n-k = s \) in the R.H.S, we have

\[ \frac{d^k}{dx^k} \sum_{n=k}^{\infty} L_n(x)t^n = \frac{d^k}{dx^k} \sum_{s=0}^{\infty} L_{s+k}(x)t^{s+k} \]

\[ = \frac{d^k}{dx^k} \sum_{n=0}^{\infty} L_{n+k}(x)t^{n+k} \]

\[ ds \frac{d^k}{dx^k} \sum_{n=k}^{\infty} L_n(x)t^n = \sum_{n=0}^{\infty} (-1)^k L_n^k(x)t^{n+k} \]

Use equation (4.24), then equation (\ast) will be in the form:

\[ \frac{d^k}{dx^k} \left[ \exp \left\{ -xt/(1-t) \right\} \right] = \sum_{n=0}^{\infty} (-1)^k L_n^k(x)t^{n+k} \]

Differentiate the L.H.S. \( k \) times we have

\[ \frac{(-1)^k}{(1-t)^{k+1}} \left[ \exp \left\{ \frac{-xt}{1-t} \right\} \right] = \sum_{n=0}^{\infty} (-1)^k L_n^k(x)t^{n+k} \]

Ten we have the required.

Similarly the students can prove the following relations:

(4.28) (i) \( L_n^k(x) = e^x x^{-k} \frac{d^n}{dx^n} \left( x^{n+k} e^{-x} \right) \)

And the following orthogonal relation:

(4.29) (ii) \( \int_0^\infty e^{-x} x^k L_n(x)L_m^k(x)dx = \frac{(n+k)!}{n!} \delta_{nm} \)

**4-6 Recurrence relations for associated Laguerre function:**

(4.30) (i) \( L_{n-1}^k(x) + L_{n-1}^k(x) = L_n^k(x) \)

**Proof.** Use equation (4.25)

\[ L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{(n+k)!x^r}{(n-r)!(k+r)!r!} \]

We have
\[ L^k_{n-1}(x) + L^{-1}_n(x) = \sum_{r=0}^{n-1} (-1)^r \left( \frac{(n-1+k)!x^r}{(n-1-r)!(k+r)!} + \sum_{r=0}^{n} (-1)^r \frac{(n+k-1)!x^r}{(n-r)!(k-1+r)!} \right) + (-1)^n \frac{(n+k-1)!x^n}{(n-n)!(k-1+n)!n!} \]

\[ L^k_n(x) + L^{-1}_{n+1}(x) = \sum_{r=0}^{n-1} (-1)^r \frac{(n+k-1)!x^r}{(n-r-1)!(k+r+1)!r!} \left\{ \frac{1}{k+r} + \frac{1}{n-r} \right\} + (-1)^n \frac{x^n}{n!} \]

\[ = (-1)^n \frac{x^n}{n!} + \sum_{r=0}^{n-1} (-1)^r \frac{(n+k-1)!x^r}{(n-r-1)!(k+r+1)!r!} \left( \frac{1}{k+r} + \frac{1}{n-r} \right) \]

\[ = (-1)^n \frac{x^n}{n!} + \sum_{r=0}^{n-1} (-1)^r \frac{(n+k)(n+k-1)!x^r}{(n-r-1)!(k+r+1)!r!(k+r)(n-r)} \]

\[ = (-1)^n \frac{x^n}{n!} + \sum_{r=0}^{n-1} (-1)^r \frac{(n+k)!x^r}{(n-r)!(k+r)!r!} \]

\[ = \sum_{r=0}^{n} (-1)^r \frac{(n+k)!x^r}{(n-r)!(k+r)!r!} = \text{R.H.S.} \]

**The second relation**

\[(4.31) \text{(ii)} \quad (n+1)L^k_{n+1}(x) = (2n+k+1-x)L^k_n(x) - (n+k)L^k_{n-1}(x) \]

**Proof.** From equation (4.17)

\[(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \]

By shifting \(n\) by \(n+k\) in the above equation, we have

\[(n+k+1)L_{n+k+1}(x) = (2n+2k+1)L_{n+k}(x) - xL_{n+k}(x) - (n+k)L_{n+k-1}(x) \]

Differentiate \(k\) times, we have

\[ (n+k+1) \frac{d^k}{dx^k} L_{n+k+1}(x) = (2n+2k+1) \frac{d^k}{dx^k} L_{n+k}(x) \]

\[ - \frac{d^k}{dx^k} \{xL_{n+k}(x)\} - (n+k) \frac{d^k}{dx^k} L_{n+k-1}(x) \]

Apply Leibniz theorem

\[ (n+k+1)L^{(k)}_{n+k+1}(x) = (2n+2k+1)L^{(k)}_{n+k}(x) - xL^{(k)}_{n+k}(x) - kL^{(k-1)}_{n+k}(x) \]

\[ - (n+k)L^{(k)}_{n+k-1}(x) \]

Use the relation

\[ L^k_n(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x) = (-1)^k L^k_{n+k}(x) \]

We have
\[ (n+k+1)(-1)^{k} L_{n+1}^{k}(x) = (2n+2k+1)(-1)^{k} L_{n}^{k}(x) - \]

Use
\[ -(-1)^{k} xL_{n}^{k}(x) - (-1)^{k-1} k L_{n+1}^{k-1}(x) - (-1)^{k} (n+k)L_{n}^{k}(x) \]

equation (4.30), we have
\[ (n+k+1) L_{n+1}^{k}(x) = (2n+2k+1) L_{n}^{k}(x) - xL_{n}^{k}(x) + k [ L_{n+1}^{k}(x) - L_{n}^{k}(x) ] - (n+k) L_{n}^{k}(x) \]

Then equation (*) take the form
\[ (n+k+1) L_{n+1}^{k}(x) = (2n+2k+1) L_{n}^{k}(x) - xL_{n}^{k}(x) + k [ L_{n+1}^{k}(x) + L_{n}^{k}(x) ] - (n+k) L_{n}^{k}(x) \]

Apply relation (4.30) again by replace every \( n \) by \( n+1 \), we have
\[ (n+k+1) L_{n+2}^{k}(x) = (2n+k+1) L_{n+1}^{k}(x) - xL_{n+1}^{k}(x) + k [ L_{n+2}^{k}(x) + L_{n+1}^{k}(x) ] - (n+k) L_{n+1}^{k}(x) \]

Then we have
\[ (n+1) L_{n+1}^{k}(x) = (2n+k+1) L_{n}^{k}(x) - xL_{n}^{k}(x) - (n+k) L_{n}^{k}(x). \]

The third equation
\[ (4.32) \text{ (iii)} \quad xL_{n}^{k}(x) = nL_{n}^{k}(x) - (n+k) L_{n}^{k}(x) \]

Use the relation (4.19)
\[ xL_{n}^{k}(x) = nL_{n}^{k}(x) - nL_{n-1}^{k}(x) \]

Shift \( n \) by \( n+k \), we have
\[ xL_{n+k}^{k}(x) = (n+k) L_{n+k}^{k}(x) - (n+k) L_{n+k-1}^{k}(x) \]

Differentiate \( k \) times
\[ \frac{d^{k}}{dx^{k}} \{ xL_{n+k}^{k}(x) \} = (n+k) \frac{d^{k}}{dx^{k}} L_{n+k}^{k}(x) - (n+k) \frac{d^{k}}{dx^{k}} L_{n+k-1}^{k}(x) \]

And use the relation
\[ (4.24) L_{n}^{k}(x) = (-1)^{k} \frac{d^{k}}{dx^{k}} L_{n+k}^{k}(x) \]

We have
\[ xL_{n}^{k}(x) + kL_{n}^{k}(x) = (n+k) L_{n}^{k}(x) - (n+k) L_{n-1}^{k}(x) \]

Then we have
\[ xL_{n}^{k}(x) = nL_{n}^{k}(x) - (n+k) L_{n-1}^{k}(x). \]

To prove the following relation:
\[ (4.33) \text{ (iv)} \quad L_{n}^{k}(x) = - \sum_{r=0}^{n-1} L_{r}^{k}(x) \]

Use the relation (4.21)
\[ L_{n}^{k}(x) = - \sum_{r=0}^{n-1} L_{r}^{k}(x) \]

Shift \( n \) by \( n+k \) and differentiate \( k \) times we have
\[
\frac{d^k}{dx^k} L'_{n+k}(x) = -\sum_{r=0}^{n+k-1} \frac{d^k}{dx^k} L_r(x)
\]

Apply the relation (4.24), we have

\[
(-1)^k L'_n(x) = -\sum_{r=k}^{n+k-1} \frac{d^k}{dx^k} L_r(x)
\]

The derivative =0 for \( r < k \):

Put \( r = k + s \) in the R.H.S., we have

\[
(-1)^k L'_n(x) = -\sum_{s=0}^{n-1} \frac{d^k}{dx^k} L_{s+k}(x)
\]

Apply the relation (4.24) again, we have

\[
L'_n(x) = -\sum_{r=0}^{n-1} L_r(x).
\]

To prove the following relation

\[
(4.34) \quad (v) \quad L^k_n(x) = -L^k_{n-1}(x)
\]

From definition of associated Laguerre function and differentiate we have

\[
L^k_n(x) = \frac{d}{dx} \sum_{r=0}^{n} (-1)^r \frac{(n+k)!x^r}{(n-r)!(k+r)!r!}
\]

\[
= \sum_{r=1}^{n} (-1)^r \frac{(n+k)!x^{r-1}}{(n-r)!(k+r)!(r-1)!}
\]

Shift \( r = 1 + s \), we found

\[
L^k_n(x) = \sum_{s=0}^{n-1} (-1)^{1+s} \frac{(n+k)!x^s}{(n-s-1)!(k+s+1)!s!}
\]

We can write the R.H.S. in the form

\[
L^k_n(x) = -\sum_{s=0}^{n-1} (-1)^s \frac{n+1+ (k+1)}{(n-1-s)(k+s+1)!s!} x^s
\]

Use the definition of associated Laguerre function, we have

\[
L^k_n(x) = -L^{k+1}_{n-1}(x)
\]

We have to prove the relation

\[
(4.35) \quad (vi) \quad L^{k+1}_n(x) = \sum_{r=0}^{n} L^k_r(x)
\]

Comparing the relations (4.34),(4.33), we have

\[
\sum_{r=0}^{n-1} L^k_r(x) = L^{k+1}_{n-1}(x)
\]

By shifting \( n \) by \( n+1 \), we have
\[ L_n^{k+1}(x) = \sum_{r=0}^{n} L_r^k(x) \]

- **Note that** in some books they define the Laguerre function in the following form
  \[
  \frac{1}{(1-t)} \exp \left\{ -xt \right\} = \sum_{n=0}^{\infty} \ell_n(x) \frac{t^n}{n!}
  \]
  Then Laguerre equation becomes
  \[ L_n(x) = \frac{1}{n!} \ell_n(x) \cdot \]

### 4-7 General examples

**Example 1.** Prove that

\[ (4.36) \int_{t}^{\infty} e^{-x} L_n^k(x) dx = e^{-t} \left[ L_n^k(t) - L_n^{k-1}(t) \right] \]

Integrate by parts the L.H.S. we have

\[ I = \left[ -e^{-t} L_n^k(x) \right]_{t}^{\infty} - \int_{t}^{\infty} (-e^{-x}) L_n^k(x) dx \]

\[ = e^{-t} L_n^k(t) + \int_{t}^{\infty} e^{-x} L_n^k(x) dx \]

Use the relation (4.33), we have

\[ I = e^{-t} L_n^k(t) - \int_{t}^{\infty} e^{-x} \left( \sum_{r=0}^{n-1} L_r^k(x) \right) dx \]

\[ = e^{-t} L_n^k(t) - \sum_{r=0}^{n-1} \int_{t}^{\infty} e^{-x} L_r^k(x) dx \]

Substitute from equation (4.36) in value of I, we have

\[ \int_{t}^{\infty} e^{-x} L_n^k(x) dx + \sum_{r=0}^{n-1} \int_{t}^{\infty} e^{-x} L_r^k(x) dx = e^{-t} L_n^k(t) \]

The L.H.S. can be written in the form

\[ (* ) \sum_{r=0}^{n} \int_{t}^{\infty} e^{-x} L_r^k(x) dx = e^{-t} L_n^k(t) \cdot \]

We can write the relation

\[ e^{-x} L_n^k(t) = \sum_{r=0}^{n} e^{-x} L_r^k(x) - \sum_{r=0}^{n-1} e^{-x} L_r^k(x) \]

And integrate w.r.to \( x \)
\[
\int \limits_{t} e^{-x} L_{n}^{k}(x) \, dx = \sum_{r=0}^{n} \int \limits_{t} e^{-x} L_{r}^{k}(x) \, dx - \sum_{r=0}^{n} \int \limits_{t} e^{-x} L_{r}^{k}(x) \, dx
\]

Use the relation (*) we have
\[
\int \limits_{t} e^{-x} L_{n}^{k}(x) \, dx = e^{-t} L_{n}^{k}(t) - e^{-t} L_{n-1}^{k}(t)
\]
\[
= e^{-t} \{ L_{n}^{k}(t) - L_{n-1}^{k}(t) \}.
\]

Example 2. Prove the following relation
\[
(4.37) \quad L_{n}^{\alpha, \beta+1}(x + y) = \sum_{r=0}^{n} L_{r}^{(\alpha)}(x)L_{n-r}^{\beta}(y).
\]

Proof. Use the definition of associated Laguerre function
\[
\frac{1}{(1-t)^{k+1}} \exp \left\{ -\frac{xt}{(1-t)} \right\} = \sum_{n=0}^{\infty} t^{n} L_{n}^{k}(x)
\]
Take the form
\[
\sum_{n=0}^{\infty} L_{n}^{\alpha, \beta+1}(x + y)t^{n} = \frac{\exp \left\{ -(x+y)t/(1-t) \right\}}{(1-t)^{\alpha+\beta+2}}
\]
Then \( L_{n}^{\alpha, \beta+1}(x + y) \) is coeff. of \( t^{n} \) in the expansion
\[
\frac{\exp \left\{ -(x+y)t/(1-t) \right\}}{(1-t)^{\alpha+\beta+2}}
\]
This expansion can be written in the form
\[
\frac{\exp \left\{ -(x+y)t/(1-t) \right\}}{(1-t)^{\alpha+\beta+2}} = \frac{\exp \left\{ -xt/(1-t) \right\} \exp \left\{ -yt/(1-t) \right\}}{(1-t)^{\alpha+1}} \frac{\exp \left\{ -yt/(1-t) \right\}}{(1-t)^{\beta+1}}
\]
\[
= \sum_{r=0}^{\infty} L_{r}^{\alpha}(x)t^{r} \sum_{s=0}^{\infty} L_{s}^{\beta}(y)t^{s}
\]
\[
= \sum_{r,s=0}^{\infty} L_{r}^{\alpha}(x) \cdot L_{s}^{\beta}(y)t^{r+s}
\]
To have coeff. of \( t^{n} \), put \( r + s = n \) where \( r \leq n \), we have
\[
t^{n} \text{ معامل } = \sum_{r=0}^{n} L_{r}^{\alpha}(x) \cdot L_{n-r}^{\beta}(y)
\]
Which complete the proof.

Example 3. Prove the following relation
\[
(4.38) \quad J_{m} \left\{ 2\sqrt{xt} \right\} = e^{-t} (xt)^{\frac{m}{2}} \sum_{n=0}^{\infty} L_{n}^{m}(x)t^{n} / (n+m)!
\]
Where \( J_{m}(y) \) is Bessel’s function of the first kind and order \( m \).

Solution. From Bessel’s function, we have
Multiply equation (4.39) in $e^{\prime}(xt)^{-\frac{1}{2}}$, we have
\[
e^{\prime}(xt)^{-\frac{1}{2}} J_m\left\{2\sqrt{xt}\right\} = e^{\prime}(xt)^{-\frac{1}{2}} \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(m+r)!} \{xt\}^{r+\frac{1}{2}}
= e^{\prime} \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(m+r)!} \{xt\}^{r}
= \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(m+r)!} x^r t^r
\]
Then we get the relation
\[
e^{\prime}(xt)^{-\frac{1}{2}} J_m\left\{2\sqrt{xt}\right\} = \sum_{r,s=0}^{\infty} (-1)^r \frac{x^r t^{r+s}}{r!(m+r)!s!}
\]
To find the R.H.S. in power $t^n$ take $s + r = n$ with condition $r \leq n$, we have
\[
e^{\prime}(xt)^{-\frac{1}{2}} J_m\left\{2\sqrt{xt}\right\} = \sum_{r=0}^{\infty} (-1)^r \frac{x^r t^n}{r!(m+r)!(n-r)!}
= \sum_{r=0}^{\infty} (-1)^r \frac{(n+m)! x^r t^n}{(n+m)! r!(m+r)!(n-r)!}
= \frac{1}{(n+m)!} L^m_n(x).
\]

**General exercises on Laguerre function**

1. **Find** $L_4$ and prove that it satisfies Laguerre differential equation at $n = 4$.
2. **Express the following functions**
   
   (i) $f(x) = 7$
   (ii) $f(x) = x$
   (iii) $f(x) = x^3 - 3x^2 + 2x$

   In Laguerre polynomials.
3. **Find** the general solution of Laguerre equation in the following cases:
   
   (i) $n = 0$
   (ii) $n = 1$
   (iii) $n = 2$
4. **Prove that**
   
   $L''_n(0) = \frac{1}{2} n(n-1)$

   And then calculate $L^{(4)}_n(0)$, $L^{(3)}_n(0)$.
5. **Prove the following relations**:
(a) \[ \int_0^\infty e^{-x^k} L_n(x) \, dx = \begin{cases} 0 & k < n \\ (-1)^n n! & k = n \end{cases} \]

(b) \[ \int_0^\infty (x-t)^m L_n(t) \, dt = \frac{m!n!}{(m+n+1)!} x^{m+1} L_n^{m+1}(x) \]

(c) \[ L_n^k(x) = (-1)^n \frac{2^{2k} k!(n+k)!}{\pi(2k)!(2n)!} \int_1^\infty (1-t^2)^{k-1/2} H_{2n}(\sqrt{x}) \, dt \]

(d) \[ n! \frac{d^m}{dx^m} \{e^{-x^k} L_n^k(x)\} = (m+n)! e^{-x^k} L_{m+n}^{k-m}(x) \]

(e) \[ \int_0^\infty e^{-x^k} \{L_n^k(x)\}^2 \, dx = \frac{(n+k)!}{n!} (2n+k+1) \cdot \]

6- **Prove the following relations:**

(a) \[ L_n^+(x) = (-1)^n \frac{1}{2^{2n+1} n! x} H_{2n+1}(\sqrt{x}) \]

(b) \[ L_n^-(x) = (-1)^n \frac{1}{2^{2n} n!} H_{2n}(\sqrt{x}) \cdot \]