On a perturbed functional integral equation of Urysohn type

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a b s t r a c t

We study the existence of monotonic solutions for a perturbed functional integral equation of Urysohn type in the space of Lebesgue integrable functions on an unbounded interval.
The technique associated with measures of noncompactness (in both the weak and the strong sense) and the Darbo fixed point are the main tool to prove our main result.

Keywords:
Functional integral equation
Existence
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Compact in measure

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NONDECREASING SOLUTIONS OF A FRACTIONAL QUADRATIC INTEGRAL EQUATION OF URYSOHN-VOLTERRA TYPE

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ABSTRACT

In this paper we study a very general quadratic integral equation of fractional order. We show that the quadratic integral equations of fractional orders has at least one monotonic solution in the Banach space of all real functions defined and continuous on a bounded and closed interval.

The concept of a measure of noncompactness related to monotonicity, introduced by J. Bana´s and L. Olszowy, and a fixed point theorem due to Darbo are the main tools in carrying out our proof.

In fact we generalize, improve the results of the paper [M.A. Darwish, On quadratic integral equation of fractional orders, J. Math. Anal. Appl. 311 (2005), 112–119]. Also, we extend and generalize the results of the paper [J. Bana´s and B. Rzepka, Monotonic solutions of a quadratic integral equation of fractional order, J. Math. Anal. Appl. 332 (2007), 1370–1378].

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Existence of monotone solutions of a perturbed quadratic integral equation of Urysohn type

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Abstract

We consider a very general quadratic integral equation and we prove the existence of monotonic solutions of this equation in $C[0;1]$. Our analysis depends on a suitable combination of the measure of noncompactness introduced by Bana´s and Olszowy and the Darabo fixed point theorem.

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